SOLUTION BY J. L. RILEY, Stephenville, Texas.

Evaluating the first determinant, we have $[R(R - 2ay^2 - 2bu^2)]^2$, where

$$R = x^2 + ay^2 + bu^2 + abv^2.$$

The second determinant, when expanded, gives $a[-2R(xy + buv)]^2$; the third gives $b[2R(avy - ux)]^2$; the fourth gives 0; and the fifth determinant gives $[-R^2]^2$. But

$$[R(R - 2ay^2 - 2bu^2)]^2 + a[-2R(xy + buv)]^2 + b[2R(avy - ux)]^2 \neq [-R^2]^2$$

unless $(ay^2)(bu^2) = 0$. Hence the relation stated in the problem does not always hold.

520 (Geometry). Proposed by ALBERT A. BENNETT, University of Texas.

On a given tangent to a circle determine a point such that, if a secant be drawn joining this point to the extremity of the diameter which is perpendicular to the given tangent, the segment of this secant exterior to the circle will be equal in length to a given segment.

SOLUTION BY A. M. HARDING, University of Arkansas.

Let us denote the radius of the given circle by r and the length of the given segment by $2d$. Let AB be the diameter perpendicular to the tangent at the point of tangency A. Take a length $AC = d$ along the tangent from A. Join BC. Take D on CB so that $CD = d$. With center at B draw arc DE cutting the circle at E. Produce BE to cut the tangent at P. Then P is the required point.

Proof:

$$BE = BD = \sqrt{d^2 + 4r^2} - d.$$

Since AE is perpendicular to BP it follows that

$$AB^2 = BE \times BP = BE(BE + EP).$$

Hence,

$$4r^2 = (\sqrt{d^2 + 4r^2} - d)(\sqrt{d^2 + 4r^2} - d + EP).$$

From this equation we find

$$EP = 2d.$$

Note: If a point Q be taken on the tangent such that $AQ = AP$, this point Q will also satisfy the conditions of the problem.

Also solved by MAY PHALOR, H. T. AUDE, HERBERT N. CARLETON, OSCAR S. ADAMS, H. C. FEEMSTER, and PAUL CAPRON.

521 (Geometry). Proposed by R. M. MATHEWS, Riverside, Cal.

A variable circle, with center on the line l and passing through a fixed point P, cuts a fixed circle in A and B. Prove that the common chord AB and the perpendicular to l through P intersect in a fixed point.

SOLUTION BY L. E. MENSENKAMP, Freeport, Illinois

It is convenient to employ rectangular coördinates. Let l be taken as the axis of x and the point P on the y-axis; then the perpendicular to l through P is the axis of y. Under these conditions, it follows from elementary analytic geometry that the equation of the variable circle is

$$(x - \alpha)^2 + y^2 = r^2,$$

where r is the radius of the variable circle. The equation of the fixed circle may be taken as

$$(x - a)^2 + (y - b)^2 = c^2.$$

Subtracting the first equation from the second, we get

$$2(\alpha - a)x - 2by = c^2 - a^2 + \alpha^2 - r^2 - b^2,$$

which is the equation of a straight line through A and B. To find the point in which this line cuts the axis of y, set x equal to zero and solve the resulting equation for y. Since $r^2 - \alpha^2$ is independent of the position of the variable circle, it follows that the intersection of the chord AB and the perpendicular to l through P is fixed.

It is interesting to note that the above proof still holds when the intersections of the two circles are imaginary points.

Also solved by H. C. Feemster, Oscar S. Adams, Harold R. Schaufler, A. M. Harding, Paul Capron, Margaret F. Willcox, and Roger A. Johnson.

522 (Geometry). Proposed by GEORGE Y. SOSNOW, Newark, N. J.

Prove that the sum of the squares of the edges of a tetrahedron is equal to four times the sum of the squares of the lines joining the middle points of the opposite edges.

Solution by R. M. Mathews, Riverside, California.

Let $ABCD$ be the tetrahedron with X, the mid-point of BD, opposite to Y, the mid-point of AC. In the triangle BYD, we have

$$2XY^2 = BY^2 + DY^2 - 2BX^2,$$

from the theorem: The sum of the squares on two sides of a triangle is equal to twice the square on half the third side plus twice the square on the median to that side.

By the same theorem, we find $2BY^2$ in the triangle ABC and $2DY^2$ in the triangle ADC. Then,

$$4XY^2 = AB^2 + BC^2 + CD^2 + DA^2 - AC^2 - BD^2.$$

Similarly with U, the middle point of AB, and V, of DC:

$$4UV^2 = AC^2 + CB^2 + BD^2 + DA^2 - AB^2 - DC^2;$$

and with T, the middle point of AD, and W, of BC:

$$4TW^2 = AC^2 + CD^2 + DB^2 + BA^2 - AD^2 - BC^2.$$

By addition, we have

$$4(XY^2 + UV^2 + TW^2) = AB^2 + BC^2 + CD^2 + AD^2 + AC^2 + BD^2.$$

Remark: Considering any one of the three equations above we have proved the theorem: Four times the square on the median joining two opposite edges of a tetrahedron is equal to the sum of the squares on the other edges minus the sum of the squares on the edges which it joins.

Also solved by Horace Olson, Oscar S. Adams, J. L. Riley, and H. C. Gossard.

345 (Mechanics). Proposed by J. L. RILEY, Northeastern State Normal School, Tahlequah, Okla.

Two particles A and B are together in a smooth circular tube. A attracts B with a force whose acceleration is ω^2 and moves along the tube with uniform angular velocity 2ω, B being initially at rest; prove that the angle φ subtended by AB at the center after a time t is given by the equation

$$\log \tan \frac{\pi + \varphi}{4} = \omega t.$$

Solution by William Hoover, Columbus, Ohio.

Let B' be the initial positions of A and B, and put $BB' = s$, and $AB = x$.

Resolving forces tangentially,

$$\frac{d^2s}{dt^2} = \omega^2 x \cos ABC, \tag{1}$$

BC being the tangent through B.

Now if O be the center of the circle,

$$\angle AOB = \frac{2r\omega t - s}{r}, \qquad \angle ABC = \frac{2r\omega t - s}{2r},$$

$$x \cos ABC = 2r \sin \tfrac{1}{2}AOB \cos ABC = 2r \sin \frac{2r\omega t - s}{2r} \cos \frac{2r\omega t - s}{2r} = r \sin \frac{2r\omega t - s}{r}$$

$$= r \sin (2\omega t - \theta), \ \theta \text{ being the angle } BOB' \text{ and } s = r\theta.$$